ISRAEL JOURNAL OF MATHEMATICS, Vol. 22, No. 1, 1975

THE IRREGULAR AND NON-HYPERREGULAR α-r.e. DEGREES

BY
RICHARD A. SHORE

ABSTRACT

We characterize those admissible ordinals α which have precisely one α-r.e. degree containing a non-regular or non-hyperregular set. For all other α we prove that any such degree $\mathbf{c}$ can be split into two strictly smaller such degrees $\mathbf{a}$ and $\mathbf{b}$ with $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$. We also prove that weak α-recursiveness ($\leqq_{w\alpha}$) is intransitive on the α-r.e. sets just in case there is more than one non-hyperregular α-r.e. degree.

Non-regular and non-hyperregular sets are two of the major sources of differences between ordinary recursion theory and recursion theory on all admissible ordinals. (Recall that $A \subseteq \alpha$ is *non-regular* if $A \cap \beta$ is not α-finite, i.e. not a member of L_α, for some $\beta < \alpha$. It is *non-hyperregular* if there is a function f weakly α-recursive in A which maps some $\beta < \alpha$ onto an unbounded subset of α. We call the α-degrees of such sets *irregular* and *non-hyperregular* respectively.) Not only do they prevent one from straightforwardly generalizing various constructions of ordinary recursion theory, but they also provide actual counterexamples to several standard theorems.

Although the pathologies of non-regularity and non-hyperregularity can arise even among the α-r.e. sets, we are somewhat better off if we restrict our attention to the α-r.e. degrees. The main advantage here is that one has various theorems that guarantee the existence of "nice" representatives from each α-r.e. degree. Thus, for example, Sacks [4] has shown that every α-r.e. degree contains a regular α-r.e. set, while Simpson [11] has proved that one can be found whose complement has order type the recursive cofinality of the degree. Indeed, results of this type have played an essential role in the proofs of most theorems about the α-r.e. degrees (e.g. [6], [8] and [11]).

Received November 14, 1974

In this paper we will continue to exploit these results together with the priority method developed in [8] and [9] to analyze the structure of the irregular and non-hyperregular α-r.e. degrees. Of course, if $\alpha^* = \alpha$ (or α is Σ_2 admissible respectively) there are no such degrees. Otherwise we are faced with a basic dichotomy. Either there is precisely one irregular (non-hyperregular) α-r.e. degree or there are many and the structure of these degrees is very rich.

In Section 2 we characterize those admissible ordinals for which the first possibility occurs and prove a sample theorem for the others:

If $\mathbf{c}$ is a non-α-recursive irregular (non-hyperregular) α-r.e. degree then there are irregular (non-hyperregular) α-r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$, $\mathbf{c} \not\leqq_\alpha \mathbf{a}$ and $\mathbf{c} \not\leqq_\alpha \mathbf{b}$. Indeed, by similar arguments one can embed in these degrees all the partial orderings that Lerman [3] embeds in the α-r.e. degrees. Moreover, the density theorem for all α-r.e. degrees [9] (together with the characterization of irregular degrees in terms of their recursive cofinality [11]) immediately establishes the density of these degrees as well. Thus, if there is more than one irregular (non-hyperregular) α-r.e. degree, the structure of such degrees seems much like that of all the α-r.e. degrees.

Our analysis of the structure of these degrees also supplies us with some interseting counterexamples to familiar theorems of ordinary recursion theory. For example, when there is precisely one irregular α-r.e. degree, the splitting theorem must fail for non-regular α-r.e. sets. Indeed, even theorems about the α-r.e. degrees involving the jump operator can fail: If there is precisely one non-hyperregular α-r.e. degree, then every incomplete α-r.e. degree has jump $\mathbf{0}'$ [10]. This, of course, contrasts with the result of Sacks that in ordinary recursion theory there is an incomplete r.e. degree with jump $\mathbf{0}''$ [7, §16]. As a final illustration of the importance of this dichotomy we show in Section 3 that $\leqq_{w\alpha}$ is intransitive (and so not equal to $\leqq_\alpha$) on the α-r.e. degrees, just in case there is more than one non-hyperregular α-r.e. degree.

Finally, we should warn the reader that although we attempt in Section 1 to give a list of all the basic definitions of α-recursion theory as well as the more unusual facts that we need, the account is necessarily sketchy. Thus the reader would be well advised first to read [8] or the alternative presentation of the methods of [8] given in [12, §4,5] before beginning this paper. In addition, several technical points on α-recursive approximations of functions can be found in the first three sections of [9]. For general background information on α-recursion theory we suggest [5] as well as [12].

1. Definitions and other preliminaries

We first summarize the standard definitions of α-recursion theory in terms of the levels L_α of Gödel's constructible universe and the usual (strict) Σ_n hierarchy of formulas. α is *admissible* if L_α satisfies the replacement axiom schema of ZF for for Σ_1 formulas. Thus we are thinking of L_α as a model of a weak set theory. All the usual set-theoretic terms (cardinal, cofinality, etc.) will therefore have their usual definitions but interpreted inside L_α.

A set $A \subseteq \alpha$ is *α-recursively enumerable* (*α-r.e.*) if it has a Σ_1 definition over L_α, while a partial function f is *partial α-recursive* if its graph is α-r.e. It is *α-recursive* if its domain is α. (Note that since there is a one-one α-recursive map of α onto L_α, it suffices for recursion theoretic purposes to restrict our attention to subsets of α and functions on α.) Of course, an $A \subseteq \alpha$ is α-recursive if its characteristic function is, while it is *α-finite* if it is a member of L_α. Finally, we say that $A \subseteq \alpha$ is *regular* if $A \cap \beta$ is α-finite for every $\beta < \alpha$.

The basic recursion theoretic fact about admissible ordinals is that one can perform Δ_1 (= α-recursive) recursions in L_α to produce α-recursive functions. Thus, for example, we can α-recursively Gödel number the α-finite sets $K_\gamma (\gamma < \alpha)$ and the Σ_0/L_α formulas with two free variables $\varphi_\varepsilon(x, y)$. This immediately gives a Gödel numbering for the α-r.e. sets, $R_\varepsilon = \{x \mid L_\alpha \models \exists\, y\varphi_\varepsilon\,(x, y)\}$, and a *standard simultaneous α-recursive enumeration* of these sets, $R_\varepsilon^\sigma = \{x \mid (\exists\, y \in L_\sigma)\varphi_\varepsilon(x, y)\}$.

We now use this enumeration to define relative recursiveness, beginning of course with an approximation: $[\varepsilon]_\sigma^C(\gamma) = \delta$ iff

$$(\exists \rho)(\exists \eta)[\langle \gamma, \delta \rho, \eta\rangle \in R_\varepsilon^\sigma \text{ and } K_\rho \subseteq C \cap \sigma \text{ and } K_\eta \subseteq (\alpha - C) \cap \sigma]$$

(we employ some α-recursive coding $\langle, \cdots, \rangle$ of n-tuples). We then say that $[\varepsilon]^C(\gamma) = \delta$ if $[\varepsilon]_\sigma^C(\gamma) = \delta$ for some σ. (Note that this makes $[\varepsilon]^C$ a possibly multivalued function.) This enables us to define the notion of *weakly α-recursive in* ($\leqq_{w\alpha}$) for a partial function f: $f \leqq_{w\alpha} C$ iff $f = [\varepsilon]^C$ for some ε (and so in particular $[\varepsilon]^C$ is single-valued). Of course, for a set B we say that $B \leqq_{w\alpha} C$ iff the characteristic function of B is weakly α-recursive in C. We now use weak α-recursiveness to define two key notions. The *recursive cofinality* of a set A (rcf A) is the least $\gamma \leqq \alpha$ such that there is an $f \leqq_{w\alpha} A$ with domain γ and range unbounded in α. A is *hyperregular* iff rcf $A = \alpha$, otherwise it is *non-hyperregular.*

Although $\leqq_{w\alpha}$ is a useful tool, we are really interested in recovering α-finite amounts of information rather than just single values. We therefore define *α-recursive in* ($\leqq_\alpha$) by saying that $B \leqq_\alpha C$ iff there is an ε such that for all α-finite sets K_γ

$$K_\gamma \subseteq B \leftrightarrow (\exists \rho)(\exists \eta)(\exists \sigma)(\langle \rho, \eta, \gamma, 0\rangle \in R_\varepsilon^\sigma$$
$$\text{and} \quad K_\rho \subseteq C \quad \text{and} \quad K_\eta \subseteq \alpha - C),$$

and

$$K_\gamma \subseteq \alpha - B \leftrightarrow (\exists \rho)(\exists \eta)(\exists \sigma)(\langle \rho, \eta, \gamma, 1\rangle \in R_\varepsilon^\sigma$$
$$\text{and} \quad K_\rho \subseteq C \quad \text{and} \quad K_\eta \subseteq \alpha - C).$$

As $\leqq_\alpha$ is transitive and reflexive, it gives us a notion of α-degree:

$$\deg(A) = \{B \mid B \leqq_\alpha A \leqq_\alpha B\}.$$

As usual, the α-degrees form an upper semi-lattice ordered by $\leqq_\alpha$.The *join* of two degrees deg $(A) \vee$ deg (B) is deg (C), where

$$C = \{2 \cdot \gamma \mid \gamma \in A\} \cup \{2 \cdot \gamma + 1 \mid \gamma \in B\}.$$

We call an α-degree *α-r.e., regular, irregular, hyperregular* or *non-hyperregular,* if it contains an α-r.e., regular, non-regular, hyperregular or non-hyperregular set, respectively. (Note that if an α-degree is (non-) hyperregular then every set in it is (non-) hyperregular. An α-degree, however, can be both regular and irregular. It is called non-regular if no member is regular.) As usual, there is a largest α-r.e. degree $\mathbf{0}'$ that of $\{\langle x, y\rangle \mid x \in R_y\}$ called the *complete α-r.e. degree.* (The complete α-r.e. sets are the ones in this degree.)

For our last set of definitions we turn to the notions of projectum and cofinality. We define the Σ_n-projectum of α (relative to A), written $\sigma n\,\mathrm{p}(\alpha)$ ($\sigma n\,\mathrm{p}_A(\alpha)$), as the least $\beta \leqq \alpha$ for which there is a one-one $\Sigma_n(\Sigma_n(A))$ map of α into β. The key fact here is that $\sigma n\,\mathrm{p}(\alpha)$ is also the least β such that there is a Σ_n/L_α subset of β which is not in L_α[2]. Note that the usual notation for the $\sigma 1\mathrm{p}(\alpha)$ is α^*, and the key fact says that any α-r.e. set bounded below α^* is α-finite. Similarly the $\Sigma_n(\Sigma_n(A))$ cofinality of $\delta \leqq \alpha$, written $\sigma n\,\mathrm{cf}(\delta)$ ($\sigma n\,\mathrm{cf}_A(\delta)$, is the least β such that there is a $\Sigma_n(\Sigma_n(A))$ map of β onto an unbounded subset of δ.

1.0. As an exercise in definition chasing note that, if C is a complete regular α-r.e. set and D is any regular α-r.e. set, then $\sigma 2\,\mathrm{cf}(\alpha) = \mathrm{rcf}\, C \leqq \mathrm{rcf}\, D = \sigma 1\,\mathrm{cf}_D(\alpha)$ and $\sigma 2\mathrm{p}(\alpha) = \sigma 1\mathrm{p}_C(\alpha) \leqq \sigma 1\mathrm{p}_D(\alpha)$. To cite yet another useful exercise, we note that $\sigma 2\,\mathrm{cf}(\alpha) = \sigma 2\,\mathrm{cf}(\mathrm{rcf}\, D) = \sigma 2\,\mathrm{cf}(\sigma 2\,\mathrm{cf}(\mathrm{rcf}\, D)) = \sigma 2\mathrm{cf}(\mathrm{cf}(\alpha^*)^{L_\alpha})$. Here one uses the admissibility of α as well as the various definitions.

We close this section with some important facts about the α-r.e. degrees:

1.1 Every α-r.e. degree is regular [4].

1.2 Every α-r.e. degree contains a *conscientious* α-r.e. set, i.e., a regular α-r.e. set C such that $\bar{C}(=\alpha - C)$ is unbounded in α and its order type, $|\bar{C}|$, is the recursive cofinality of C[11].

1.3. An α-r.e. degree is irregular iff $\alpha^* < \alpha$ and its recursive cofinality is $\leqq \mathrm{cf}(\alpha^*)^{L_\alpha}$ (the cofinality of α^* in L_α) [11]. (Note that rcf is an invariant of α-degree since $f \leqq_{w\alpha} A \leqq_\alpha C$ implies that $f \leqq_{w\alpha} C$.)

1.4. If B is a regular α-r.e. set and $\sigma 1\mathrm{p}_B(\alpha) > \mathrm{rcf}(B)$, then $\mathbf{0}' \leqq_\alpha B$ (Lemma 3.3 of [9]).

Finally a fact about projecta:

1.5. If B is a regular α-r.e. set such that $\sigma 1\mathrm{p}_B(\alpha) \leqq \mathrm{rcf}(B) = \beta$, then there is a *tame* onto map $f:\beta \to \alpha$ (that is f can be approximated by an α-recursive function of two variables $f^\sigma(x)$ such that

$$(\forall \delta < \beta)(\exists \tau)\ (\forall \sigma > \tau)(\forall x < \delta)(f^\sigma(x) = f(x)).$$

Such a function f is called a *tame Σ_2 projection* of β onto α. The usefulness of such functions lies in the fact that they can be used in priority arguments to put the requirements in a short (i.e. β) list. [Such a function is given by the approximation to $\{\varepsilon\}_\sigma^{B^\sigma}(x)$ given in Section 1 of [9] where $f = [\varepsilon]^B$. The tameness follows immediately from the definition of rcf B and the assumption that $\sigma 1\mathrm{p}_B(\alpha) \leqq$ rcf B.] Indeed, it was for just this reason that tame Σ_2 functions were introduced by Lerman [3].

2. The main theorem

Our goal is to characterize those admissible ordinals α that have precisely one irregular (non-hyperregular) α-r.e. degree, and to prove a splitting theorem for these degrees for all other α.

THEOREM 2.1. *There is precisely one irregular (non-hyperregular) α-r.e. degree if and only if $\sigma 2\mathrm{p}(\alpha) > \mathrm{cf}(\alpha^*)^{L_\alpha}$ ($\sigma 2\,\mathrm{cf}(\alpha) < \alpha$ and $\sigma 2\,\mathrm{p}(\alpha) > \mathrm{rcf}\, D$ for every non-hyperregular α-r.e. set D). Moreover, if this condition fails, then for every irregular (non-hyperregular) α-r.e. degree $\mathbf{c}$ there are strictly smaller irregular (non-hyperregular) α-r.e. degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \vee \mathbf{b} = \mathbf{c}$.*

PROOF. If the condition holds, then clearly there is an irregular (non-hyperregular) α-r.e. degree, as $\alpha \geqq \sigma 2\mathrm{p}(\alpha) > cf(\alpha^*)^{L_\alpha}$ implies that $\alpha > \alpha^*$, and so there are non-regular α-r.e. sets ($\sigma 2\,\mathrm{cf}(\alpha) < \alpha$ implies that $\mathbf{0}'$ is not hyperregular). Moreover, 1.1, 1.3 and 1.4 show that every irregular (non-hyperregular) α-r.e. degree must be complete.

If, on the other hand, the condition fails and there is an irregular (non-hyperregular) α-r.e. set, we claim that we may choose an α-r.e. set D such that $\alpha > \mathrm{cf}(\alpha^*)^{L_\alpha} = \mathrm{rcf}\, D \geqq \sigma\, 1p_D(\alpha)(\alpha > \mathrm{rcf}\, D \geqq \sigma\, 1p_D(\alpha))$. That there is an α-r.e. D with rcf $D = \mathrm{cf}(\alpha^*)^{L_\alpha}$ follows from Theorem 3.9 of [11] and our remarks in 1.0. If, however, $\mathrm{rcf}\, D < \sigma\, 1p_D(\alpha)$, then D is complete by 1.4 and so, again by 1.0, $\sigma\, 1p_D(\alpha) = \sigma 2p(\alpha) > \mathrm{cf}(\alpha^*)^{L_\alpha}$, contradicting our assumption that the condition failed. (The failure of the condition guarantees the existence of a non-hyperregular α-r.e. set D such that $\alpha > \mathrm{rcf}\, D \geqq \sigma 2p(\alpha)$. If D is complete, then $\sigma 2p(\alpha) = \sigma\, 1p_D(\alpha)$ by 1.0, while if D is incomplete, $\mathrm{rcf} D \geqq \sigma 1p_D(\alpha)$ by 1.4.) We will now use such a D and the associated projection guaranteed by 1.5 to prove the splitting theorem asserted above. Note that our proof will be the same for both the irregular and the non-hyperregular α-r.e. degrees. Of course, it clearly implies that there is more than one irregular (non-hyperregular) α-r.e. degree as required.

Let $\mathbf{c}$ be the given irregular (non-hyperregular) α-r.e. degree. By 1.2 we may choose a conscientious C in $\mathbf{c}$. We let c be a one-one α-recursive function enumerating C and approximate C by $C^\sigma = \{c(i) \mid i < \sigma\}$. Our goal is to construct α-r.e. sets A and B with $|\bar{A}| = \mathrm{rcf}\, C = |\bar{B}|$ if $\sigma\, 1\, p_C(\alpha) \leqq \mathrm{rcf}\, C$ or with $|\bar{A}| = \mathrm{rcf}\, D = |\bar{B}|$ otherwise. We will also want to insure that $A, B \leqq_\alpha C$, $C \not\leqq_\alpha A$, $C \not\leqq_\alpha B$ and $A \cup B = C$. As this last requirement implies that $C \leqq_\alpha A \vee B$, all of this together with 1.3 (definition of hyperregular) will establish our theorem.

Before describing the actual construction we must fix our notation. Let $\beta = \mathrm{rcf}\, C$ if $\mathrm{rcf}\, C \geqq \sigma\, 1\, p_C(\alpha)$ and let it be rcf D otherwise. Let $g: \beta \to \alpha$ be the map guaranteed by 1.5 with its tame recursive approximation given by g^σ. We can also fairly easily get a map f of $\kappa = \mathrm{rcf}\, C = |\bar{C}|$ onto an unbounded subset of α which is strictly increasing with a tame recursive approximation f^σ (see, for example, Section 3 of [9]). Note that g as well as f is α-recursive in C even if $\beta = \mathrm{rcf}\, D$, since this can occur only if $\mathrm{rcf}\, C < \sigma\, 1\, p_C(\alpha)$, which by 1.4 implies that C is complete. Similarly we let $h: \gamma \to \beta$ be an increasing unbounded function on $\gamma = \sigma 2\mathrm{cf}(\alpha) = \sigma 2\mathrm{cf}(\beta)$ with its tame recursive approximation given by h^σ.

Here one should note that $\kappa = \gamma$ if C is complete and otherwise $\kappa = \beta$.

THE GENERAL PLAN. As we enumerate an element $c(\sigma)$ in C at stage σ, we will immediately put it into A or B. This will insure that $A \cup B = C$. There will of course be negative requirements as in the usual splitting theorem [8] that will dictate via a priority scheme which set gets $c(\sigma)$. These requirements will try for each ε to preserve computations of more and more of $\bar{C}$ from A or B via ε. As usual, once ε has highest priority one sees that these preservations must be bounded or C would be α- recursive. As there are at most β elements in $\bar{C}$ all together, we will in fact have to preserve fewer than β such computations for each ε. As we have also put the reduction procedures in a β list, we can hope to make $\bar{A}$ and $\bar{B}$ small (i.e. of order type $\leqq \beta$) without overrunning the negative requirements if each such requirement also keeps out fewer than β many elements. Should we succeed, however, in making $|\bar{A}| = \beta = |\bar{B}|$, this added restriction will not make us miss any possible correct computations from A or B. We in fact assure that $\bar{A}$ and $\bar{B}$ are small by adding new positive requirements that try to put elements of C into both A and B. (Thus we give up the conclusion of [8] that $A \cap B = \emptyset$.)

To guarantee that injuries caused by initial segments of the requirements don't become unbounded in α, we use the blocking technique introduced in [8] to govern the interaction of negative requirements. We will moreover fix the blocking in advance into γ many pieces which are generated by h^σ at stage σ. The positive requirements, however, will be put in a κ list. The idea is that the δth positive requirement (for $\delta < \kappa$) will try to put all elements of $C \cap f(\delta)$ into A and B. Their interaction with the negative requirements will be determined by the β listing of reduction procedures if $\kappa = \beta$ (by the blocking into γ pieces if $\kappa = \gamma$). (We will indicate differences between these two cases by putting the case $\kappa = \gamma$ in parenthesis.) The point is, that to get $A, B \leqq_\alpha C$ we will want to know α-recursively in C what is the highest priority ever given to a particular element of C by these new requirements.

THE REQUIREMENTS. As almost everything is symmetric with respect to A and B, we will usually describe only the A part of the construction and proof. When necessary we will distinguish by subscripting with A or B to indicate any differences. We first describe the formation of the negative requirements. Our terminology will be defined simultaneously with the construction.

At stage σ we find for each $\delta < \gamma$ the least $x \not\in C^\sigma$ for which there is no negative δ requirement for x associated with a δ-active reduction procedure. If there is a δ-active reduction procedure $\varepsilon < h^\sigma(\delta)$ for which $[g^\sigma\varepsilon]_\sigma^{A^\sigma}(x) = 0$ via a computation requiring less than β many elements to be outside of A^σ (the elements enumerated in A by stage σ), we take the least such computation and *create a negative δ requirement for x associated with ε*. This requirement consists of the elements assumed to be outside of A^σ by the chosen computation. If at any later stage we put any element of the requirement into A we *destroy* it. We also destroy it at any later stage at which $g^\sigma\varepsilon$ changes value or ε is δ-inactive (because of another requirement) and x enters C. If a negative requirement is never destroyed, it is called *permanent*. Finally we call a reduction procedure $\varepsilon < \beta$ *δ-active* at stage σ unless there is a δ requirement (as yet understroyed) for some x associated with ε such that $x \in C^\sigma$, in which case it is *δ-inactive*. The idea is that, as long as we seem to have a computation showing that $[g\varepsilon]^A \neq C$, we need pay no further attention to ε.

At stage σ we also *create a positive η requirement* for each $\eta < \kappa$ and each $x < f^\sigma(\eta)$ in C^σ for which there is no such requirement.

THE CONSTRUCTION. At stage σ we first form requirements as indicated above. We then put $c(\sigma)$ into A or B. The choice is made so as to preserve as much as possible. More precisely, we consider the sets $I_A(I_B)$ of negative A (B) requirements which would be destroyed by putting $c(\sigma)$ into A (B). Let $\nu_A(\nu_B)$ be the least ordinal $\nu < \gamma$ such that $I_A(I_B)$ contains a negative ν-requirement. If $\nu_A \leqq \nu_B$, we put $c(\sigma)$ into B, otherwise it goes into A. We now proceed for each $\eta < \kappa$ in turn to put elements with positive η requirements into A(and B), unless they are in an as yet understroyed negative $\delta_A(\delta_B)$-requirement associated with an $\varepsilon < \eta$ (such that $\delta < \eta$ if $\kappa = \gamma$). We of course destroy other negative requirements as necessary for each η before going on to $\eta + 1$.

THE PRIORITY ARGUMENT. Our main goal is to show that there are less than β permanent negative δ-requirements for each $\delta < \gamma$. We will then be able to conclude that $C \not\leqq_\alpha A$ and $|\bar{A}| \leqq \beta$. A further argument will then be adduced to show that $A \leqq_\alpha C$ and so complete the proof of Theorem 2.1

LEMMA 2.2. *For each $\delta < \gamma$ there are less than β permanent negative δ'-requirements for each $\delta' < \delta$. Moreover, there is a bound on the stages at which all δ' requirements ($\delta' < \delta$) are created, destroyed or have any of their elements enumerated in C.*

PROOF. We proceed by induction on δ. Note first that if τ is the bound of the theorem, then no δ_A-requirement created after stage τ can be destroyed by an element $c(\sigma)$ via the first part of the construction at stage σ. Next let $\tau_1 \geqq \tau$ be such that $h^\sigma(\eta) = h(\eta)$ for $\sigma > \tau_1$ and $\eta \leqq \delta$, and such that $g^\sigma(\eta) = g(\eta)$ for $\eta < h(\delta)$ and $f^\sigma(\zeta) = f(\zeta)$ for $\zeta \leqq h(\delta)$ ($\zeta < \delta$ if $\kappa = \gamma$) and $\sigma > \tau_1$. Of couse, after stage τ_1 no negative δ_A requirement can be destroyed by a change in g^σ. Finally, because of the regularity of C and the admissibility of α, we can choose a $\tau_2 \geqq \tau_1$ such that

$$C^\sigma \cap fh(\delta) = C \cap fh(\delta) \quad (C^\sigma \cap f(\delta) = C \cap f(\delta)) \quad \text{for} \quad \sigma > \tau_2.$$

Now no new positive η requirements can be created after stage τ_2 for $\eta < h(\delta)$ $(\eta < \delta)$.

We now proceed by induction on $\eta < h(\delta)$ to bound the stages at which positive η requirements suceed in putting elements into A. As our procedure will be α-recursive in C and $h(\delta) < \mathrm{rcf}(C)$, the whole process will be bounded below α. ($\tau_2 + 1$ is already such a bound if $\kappa = \gamma$. All δ' requirements for $\delta' < \delta$ are now permanent and so any positive requirement which did not succeed at stage τ_2 is being thwarted by a permanent negative requirement of higher priority. It can therefore never succeed at any later stage.) Of course, those elements belonging to negative δ'-requirements for $\delta' < \delta$ never get put in A since these requirements are permanent. Suppose that we have found (α-recursively in C) a bound on the stages at which positive η' requirements put elements into A for $\eta' \leqq \eta$. From this stage on a negative δ_A-requirement, say for x, which is associated with $\eta' \leqq \eta$ can be destroyed only by η' becoming δ_A-inactive and x then entering C. We can ask α-recursively in C which x's with such requirement ever get into C. If there are none, then every δ_A-negative requirement associated with $\eta' \leqq \eta$ is permanent. Thus any positive $\eta + 1$ requirement which has not yet succeeded never will. (It must be in a permanent negative requirement of higher priority or it would get into A now.) If there are such x's, we go (α-recursively in C) to a stage by which all of them have entered C. At this stage all negative δ_A-requirements associated with $\eta' < \eta$ are permanent and so, as before, any positive $\eta + 1$ requirement not succeeding now is being thwarted by a permanent negative requirement of higher priority. To complete our induction, first note that for limit ordinals η we just begin this process above all bounds for $\eta' < \eta$. We let τ_3 be the supremum of these bounds for $\eta < h(\delta)$. ($\tau_3 = \tau_2 + 1$ if $\kappa = \gamma$.)

After stage τ_3 a negative δ_A-requirement for x associated with η can be destroyed only if η becomes δ_A-inactive and then x later enters C. To bound these stages consider first the set

$$W = \{\eta < h(\delta) \mid (\exists \sigma > \tau_3)(\eta \text{ is } \delta_A\text{-inactive at stage } \sigma)\}.$$

W is a Σ_1 subset of $h(\delta) < \beta < \alpha^*$, and so α-finite. By the admissibility of α there is then a bound τ_4 on the stages at which elements enter W by becoming δ_A-inactive. Thus any $\eta < h(\delta)$ which is δ_A-inactive at stage τ_4 remains so forever, while no $\varepsilon < h(\sigma)$ can become δ_A-inactive at any stage after τ_4. It therefore suffices to wait until a stage τ_5 by which all x's in C for which there are δ_A requirements at stage τ_4 have been enumerated in C. (Again such a stage exists by the regularity of C.) Clearly, after stage τ_5 no δ_A-requirement is ever destroyed.

We can now show that there are less than β permanent δ_A requirements. For $\eta \in W$ note that at stage τ_5 there must be fewer than β many x's for which we can have δ_A requirements associated with η, since any such x is in fact not in C while $|\bar{C}| = \beta$ and so any bounded segement of $\bar{C}$ has order type $< \beta$. Moreover, η remains δ_A-inactive forever, and so no more δ_A-requirements can be created which are associated with η.

Next note that, for any δ_A negative requirement for some x associated with an η not in W at a stage $\sigma > \tau_5$, we must have $x \not\in C$ by definition of W and the permanency of such a requirement. By the construction there can be at most one such requirement for each x. As $|\bar{C}| = \beta$, the only way to get β many requirements is to eventually form one for every $x \in \bar{C}$ (we always try for the least x for which we have no such requirement as yet). If this happens, however, we can enumerate $\bar{C}$ α-recursively as the set of elements x for which there is a δ_A negative requirement at a stage $\delta > \tau_5$. This would contradict the fact that C is not α-recursive. Thus we have shown that there are fewer than β many permanent δ_A negative requirements.

After stage τ_5 these requirements are created for x's on an initial segment of $\bar{C}$ (ignoring those with such requirements at τ_5). As $|\bar{C}| = \beta$, this is a proper initial segment and so by the regularity of C an α-finite set. Thus by the admissibility of α there is a bound τ_6 on the stages at which such requirements are created. Finally, as any requirement created by stage τ_6 consists of elements less than τ_6, the regularity of C again assures us that there is a bound τ_7 on the stages at which elements of such requirements are enumerated in C.

We have thus established the conclusions of the lemma for δ_A negative requirements. The same argument beginning now at stage τ_7 establishes the results for the δ_B negative requirements. This then carries our induction one step forward. To see that all is well at limit ordinals δ, note that the map from $\delta' < \delta$ to the bound claimed in the lemma is a Σ_2 function. As $\delta < \gamma = \sigma 2\,\mathrm{cf}(\alpha)$, the range of this function on δ is bounded. This then gives us the bound on all δ' requirements for $\delta' < \delta$ at limit ordinals δ and alows us to carry out the above argument for δ-requirements and continue the induction. ∎

We can now prove that A and B have the desired properties.

LEMMA 2.3. *$\|\bar{A}| \leqq \beta$ and $|\bar{B}| \leqq \beta$.*

PROOF. For any $\delta < \gamma$ let τ be the bound given by Lemma 2.2 for δ.

The positive requirements clearly guarantee that all the elements of $C \cap fh(\delta)$ (if $\kappa = \gamma$ replace fh by f in this proof) which are not put in A and B are kept out by some negative δ'-requirement, with $\delta' < \delta$ at stage τ. As there are less than β of them and each one keeps out fewer than β elements, we see that $A \cap fh(\delta)$ and $B \cap fh(\delta)$ agree with $C \cap fh(\delta)$, except possibly on some set of size less than β. As $\bar{C} \cap fh(\delta)$ has size less than β, so too do $\bar{A} \cap fh(\delta)$ and $\bar{B} \cap fh(\delta)$. As $f \circ h$ on γ is unbounded in α, our conclusion follows from the fact that β is a cardinal in L_α. ■

LEMMA 2.4. *$C \not\leq_\alpha A$ and $C \not\leq_\alpha B$.*

PROOF. If not, suppose for definiteness that $[g\varepsilon]^A$ is the characteristic function of C. Let $\delta < \gamma$ be such that $\varepsilon < h(\delta)$. Let τ be the bound given by Lemma 2.2 for $\delta + 1$ and let x be the least element of $\bar{C}$ for which there is no negative δ_A requirement associated with a δ_A-active reduction procedure at stage τ. As $[g\varepsilon]^A$ is the characteristic function of C, there is a computation showing that $[g\varepsilon]^A(x) = 0$ and so (as A is α-r.e.) a stage $\sigma > \tau$ at which $[g\varepsilon]^{A^\sigma}_\sigma(x) = 0$. Now if ε were δ_A-inactive at stage σ, it would be because of a computation $[g^{\sigma'}\varepsilon]^{A^{\sigma'}}_{\sigma'}(x') = 0$ for some $\sigma' < \sigma$ and $x' \in C$ which has not yet been destroyed by stage σ. Our choice of τ then guarantees that this requirement is never destroyed and so $g^{\sigma'}\varepsilon = g\varepsilon$ and the information used about $\bar{A}$ is correct and so $[g\varepsilon]^A(x') = 0$. This of course contradicts our assumption that $[g\varepsilon]^A$ is the characteristic function of C. Thus ε is δ_A-active at stage σ and so we must create a negative δ_A requirement for x at stage σ. This, however, contradicts the choice of τ as $\sigma > \tau$. ■

We now complete the proof of our theorem by establishing the last condition on A and B.

LEMMA 2.5. *$A \leq_\alpha C$ and $B \leq_\alpha C$.*

PROOF. We show that for $\eta < \kappa$ we can inductively compute $A \cap f(\eta)$ α-recursively in C and also decide which negative requirements associated with η are permanent. First recall that initial segments of f can be computed from C. Now suppose that we have computed $A \cap f(\mu)$ for every $\mu < \eta$ and we wish to decide if a negative requirement N associated with some $\eta' < \eta$ is permanent. It can be destroyed by a change in $g^\sigma\eta'$, but we can compute g restricted to η α-recursively in C and so can check this possibility. Next it could be destroyed by the x for which we have this requirement entering C after η' is inactive. To check this possibility we just ask if $x \in C$, and if it does we wait until $x \in C^\sigma$ to see if the requirement is destroyed. Finally, it can be destroyed by an element $y \in N$ entering A. This can happen either at a stage σ at which $y = c(\sigma)$ is put into A via the first part of the construction, or by y being put into A by a positive requirement. To check if the first possibility occurs we just wait until $C^\sigma \cap N = C \cap N$. The second alternative can occur only if y has a positive μ-requirement with $\mu \leqq \eta'(\mu \leqq \delta$ where δ is the least ordinal $< \gamma$ such that $\eta' < h(\delta)$ if $\kappa = \gamma$), i.e. $y \in C \cap f(\mu)$ Thus the inductive hypothesis that we have computed $A \cap f(\mu)$ for $\mu \leqq \eta' < \eta$ (note that $\delta \leqq \eta'$ if $\kappa = \gamma$) allows us to check this final possibility as well. Of course, if none of these happen N is permanent.

To now compute $A \cap f(\eta)$, it suffices to show that every element x of $C \cap (f(\eta) - \cup \{f(\mu) \mid \mu < \eta\})$ is put into A unless it belongs to a permanent negative requirement associated with a $\mu < \eta$ ($\mu < h(\delta)$ where δ is the least ordinal $< \gamma$ such that $\eta < h(\delta)$ if $\kappa = \gamma$). To see that this suffices, note that we just have to wait until each such x is put into either A or a permanent negative requirement associated with such a μ which we can recognize α-recursively in C. To actually establish the claim, consider a stage σ at which $x \in C^\sigma \cap f^\sigma(\eta)$ and such that all negative requirements associated with any $\mu < \eta$ are permanent (such a stage exists by Lemma 2.2). If x is not a member of any such negative requirement, it is clearly put into A by the second part of the construction. On the other hand, if x is in such a requirement it can never be put into A, since this would contradict the permanency of the requirement. Finally, one should note that Lemma 2.2 guarantees that for any $\eta < \beta$ this entire procedure (for every x in the set) involves only a bounded set of stages and so only α-finitely much of C is needed for the entire inductive computation. ∎

REMARK. We first proved this theorem for irregular degrees by a somewhat different method in trying to answer the question of when one can split a non-regular α-r.e. set into two disjoint α-r.e. sets of strictly lower α-degree. Of course, when there is only one irregular α-r.e. degree this is impossible. What we proved, however, was that, if $C \subseteq \alpha^*$ is a simple α-r.e. set (i.e. $\bar{C}$ is unbounded in α^* and $C \cap K \neq \emptyset$ for every α-finite K which is unbounded in α^*) and the conditions of the theorem fail, then there are disjoint simple α-r.e. sets A and B such that $A \cup B = C$, $A \leqq_\alpha C$, $B \leqq_\alpha C$, $C \not\leqq_\alpha A$ and $C \not\leqq_\alpha B$. The proof involved adding positive requirements to make both A and B simple rather than to make their complement thin. As every irregular α-r.e. degree contains a simple α-r.e. subset of α^* [11], this also proves Theorem 2.1 for the irregular α-r.e. degrees. Simpson then proved that, when the condition of

Theorem 2.1 for non-hyperregular α-r.e. degrees fails, there are incomparable non-hyperregular α-r.e. degrees (the other direction is a consequence of 1.4). Upon being informed of this result, we proved Theorem 2.1 as it now stands.

3. Weak α-recursiveness

When Driscoll [1] proved that weak α-recursiveness is not transitive on the α-r.e. sets for $\alpha = \omega_1^{CK}$, the question naturally arose as to when $\leqq_{w\alpha}$ is transitive on these sets. Stillwell [13] thought that this should occur only when α is Σ_2-admissible (an obviously sufficient condition). Simpson [11], however, found a counter example by showing that $\aleph_\omega^L$ has only one non-hyperregular α-r.e. degree, that of $\mathbf{0}'$ but was unable to settle the question in general. (The point here is that if an α-r.e. set A is hyperregular and $B \leqq_{w\alpha} A$, then $B \leqq_\alpha A$ while every α-r.e. set is α-recursive in a complete one. Thus $\leqq_{w\alpha} = \leqq_\alpha$ on the α-r.e. sets and of course $\leqq_\alpha$ is transitive.) We show that this example is indeed typical and characterize those admissible α for which $\leqq_{w\alpha}$ is intransitive in terms of the conditions of Theorem 2.1 by proving the following:

THEOREM 3.1. *$\leqq_{w\alpha}$ is intransitive on the α-r.e. sets if and only if there is an incomplete (and so more than one) non-hyperregular α-r.e. degree.*

PROOF. As noted above, the only if direction is immediate. Assume therefore that B is an incomplete α-r.e. set with rcf $B = \beta < \alpha$. Let $h: \beta \to \alpha$ have unbounded range and be weakly α-recursive in B. By [5, §25] we can let A be a complete α-r.e. set such that $(X)\ (A \leqq_{w\alpha} X \to A \leqq_\alpha X)$. We then let $\varphi(x, y)$ be a Δ_0 formula such that $y \in A \leftrightarrow \exists x \varphi(x, y)$. Finally, we choose our set C to be $\{\langle \delta, y\rangle \mid (\exists x < h(\delta))\varphi(x, y)\}$.

We first note that C is α-r.e. since

$$C = \{\langle \delta, y\rangle \mid (\exists \sigma)(\exists x < h^\sigma(\delta))\varphi(x, y)\},$$

where h^σ is the α-recursive approximation to h given in [9, §3]. Next we claim that $C \leqq_{w\alpha} B$. As C is α-r.e., there is automatically a procedure listing $\langle \delta, y\rangle \in C$ while $\langle \delta, y\rangle \notin C \leftrightarrow (\exists \gamma)\ (\gamma = h(\delta)$ and $\neg(\exists x < \gamma)\varphi(x, y))$.

(As $h \leqq_{w\alpha} B$, this can easily be converted into the statement required in the definition of $C \leqq_{w\alpha} B$.) Finally, we show that $A \leqq_{w\alpha} C$:

$$y \notin A \leftrightarrow (\beta \times \{y\}) \cap C = \varnothing.$$

(Again, as A is α-r.e., the part of the definition of $\leqq_{w\alpha}$ for $y \in A$ is automatic.) Thus $A \leqq_{w\alpha} V \leqq_{w\alpha} B$ and so the transitivity of $\leqq_{w\alpha}$ would imply tha $A \leqq_{w\alpha} B$.

Our choice of A then says that $A \leqq_\alpha B$, contradicting the incompleteness of B. ∎

References

1. G. Driscoll, *Meta-recursively enumerable sets and their metadegrees*, J. Symbolic Logic **33** (1968), 389-411.

2. R. B. Jensen, *The fine structure of the constructable hierarchy*, Ann. Math. Logic **4** (1972), 229–308.

3. M. Lerman, *On suborderings of the α-recursively enumerable α-degrees*, Ann. Math. Logic **4** (1972), 415–443.

4. G. E. Sacks, *Post's problem, admissible ordinals and regularity*, Trans. Amer. Math. Soc. **124** (1966), 1–24.

5. G. E. Sacks, *Higher Recursion Theory*, Springer-Verlag, to appear.

6. G. E. Sacks and S. G. Simpson, *The α-finite injury method*, Ann. Math. Logic **4** (1972), 343-368.

7. J. R. Shoenfield, *Degrees of Unsolvability*, North-Holland, 1971.

8. R. A. Shore, *Splitting an α-recursively enumerable set*, Trans. Amer. Math. Soc. **204** (1975), 65–78.

9. R. A. Shore, *The recursively enumerable α-degrees are dense*, Ann. Math. Logic, to appear.

10. R. A. Shore, *On the jump of the recursively enumerable α-degrees*, Trans. Amer. Math. Soc., to appear.

11. S. G. Simpson, *Admissible ordinals and recursion theory*, Ph. D. Thesis, M.I.T., 1971.

12. S. G. Simpson, *Degree theory on admissible ordinals*, in Generalized Recursion Theory, Proc. of the 1972 Oslo Symposium, J. E. Fenstad and P. G. Hinman, eds., North-Holland, 1974, pp. 165-193.

13. J. Stillwell, *Reducibility in generalized recursion theory*, Ph. D. Thesis, M.I.T., 1970.

Department of Mathematics
Cornell University
Ithaca, N.Y. 14853 U.S.A.